

Exam T3 - Thermal Physics II, Prof. G. Palasantzas

- Date: 11-12-2025
- Total number of points 100
- 10 points for taking the exam



Problem 1 (20 points)

Consider a system where its energy E behaves like $E = \alpha|x|^m$ with $\alpha > 0$ and m a positive integer (≥ 1). Calculate the average energy $\langle E \rangle$ as a function of the temperature T and the exponent m .

You could consider the relations: $\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$ and $\Gamma(z+1) = z\Gamma(z)$.

Problem 2 (25 points)

A system of N distinguishable particles with fixed volume V is at thermal equilibrium at temperature T . Each particle has three internal energy levels $E_1 = \varepsilon$, $E_2 = 2\varepsilon$ and $E_3 = 3\varepsilon$, with degeneracies $g(E_1) = g(E_3) = 1$, $g(E_2) = 2$.

(a: 5 points) Write properly the internal partition function Z_{int} of a single particle.

(b: 10 points) Calculate the internal energy U of the system of the N particles

(c: 10 points) Show that the heat capacity $C_V = \partial U / \partial T$ is given by $C_V = 2Nk_B [x^2 e^{-x} / (1 + e^{-x})^2]$ with $x = \beta\varepsilon$.

Problem 3 (20 points)

Consider the total partition function Z_N of a classical ideal gas of N ($\gg 1$) indistinguishable particles within a volume V at temperature T , and $Z_1 (= V/\lambda_{\text{th}}^3)$ the single-particle partition function due to translational motion with λ_{th} the thermal wavelength.

(a: 5 points) Write the correct form of the partition function Z_N due to translational motion and discuss briefly the regime of its validity.

(b: 5 points) Prove that chemical potential is given by $\mu = k_B T \ln[N/Z_1]$.

(c: 5 points) Is the chemical potential μ positive or negative?. Justify your answer.

(d: 5 points) Prove that the grand potential $\Phi_G = F - \mu N$ is given by $\Phi_G = -PV$.

Problem 4 (10 points)

Assume that the system of the N particles in Problem 2 can be treated as a classical ideal gas with respect to translational motion within the volume V .

(a 5 points) Write properly the total partition function of the system to include both the contribution of internal degrees of freedom and the translational motion.

(b 5 points) Calculate the heat capacity C_{Total} of the system. For the translational motion use the partition function of a classical ideal gas from Problem 3.

Problem 5 (15 points)

(a 5 points) What is the value of chemical potential of a photon gas at thermal equilibrium?

(b 10 points) Using the partition function of a harmonic oscillator $Z_{\omega} = e^{-\hbar\omega\beta/2} / (1 - e^{-\hbar\omega\beta})$ and the density of states $g(\omega) d\omega = [V\omega^2/\pi^2 c^3] d\omega$ for a photon gas in volume V at thermal equilibrium, show that the partition function Z of the photon gas is given by

$$\ln Z = -\frac{V}{\pi^2 c^3} \int_0^{\infty} \omega^2 \ln(1 - e^{-\hbar\omega\beta}) d\omega$$

Problem 1

Take $E=a|x|^n$ and consider $x \geq 0$ since the absolute value makes E an even function of x so that the integration from $-\infty$ to $+\infty$ can be twice that from 0 to $+\infty$.

$dE=na x^{(n-1)}dx=n(E/x)dx$. Thus we have $dx=(1/n)a^{-1/n}E^{(1/n-1)}dE$.

$$\langle E \rangle = \frac{\int_0^\infty E e^{-\beta E} dx}{\int_0^\infty e^{-\beta E} dx} = \frac{\int_0^\infty E^{\frac{1}{n}} e^{-\beta E} dE}{\int_0^\infty E^{\frac{1}{n}-1} e^{-\beta E} dE} = k_B T \frac{\Gamma(1 + \frac{1}{n})}{\Gamma(\frac{1}{n})} = \frac{k_B T}{n}.$$

$\underbrace{\beta^{-(\frac{1}{n}+1)} \int_0^\infty x^{(\frac{1}{n})} e^{-x} dx = \beta^{-(\frac{1}{n}+1)} \Gamma(1 + \frac{1}{n})}$
 $\underbrace{\beta^{-(\frac{1}{n})} \int_0^\infty x^{(\frac{1}{n}-1)} e^{-x} dx = \beta^{-(\frac{1}{n})} \Gamma(\frac{1}{n})}$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

$$\Gamma(1+1/n) = (1/n)\Gamma(1/n)$$

Replace n with m and you have your answer $\langle E \rangle = k_B T/m$

Problem 2

a) T_0 He $E_1 = 0$
 $Z(T) = 1 + 2e^{-\beta\epsilon} + e^{-\beta 2\epsilon} \Rightarrow$
 $Z_{int}(T) = (1 + e^{-\beta\epsilon})^2$

b) $U(T) = - \frac{d \ln Z_{int}^N}{d\beta} \Rightarrow$
 $U(T) = 2\epsilon N \frac{e^{-\beta\epsilon}}{1 + e^{-\beta\epsilon}}$

c) $C_V = \frac{\partial U}{\partial T} = \frac{d\beta}{dT} \frac{\partial U}{\partial \beta}$

$$\frac{d\beta}{dT} = -k\beta^2 \quad (1)$$

$$\frac{\partial U}{\partial \beta} = N(2\epsilon) \left[\frac{-\epsilon e^{-\beta\epsilon}}{1 + e^{-\beta\epsilon}} + \frac{\epsilon e^{-2\beta\epsilon}}{(1 + e^{-\beta\epsilon})^2} \right] \Rightarrow$$

$$\frac{\partial U}{\partial \beta} = -2N\epsilon^2 \frac{e^{-\beta\epsilon}}{(1 + e^{-\beta\epsilon})^2} \quad (2)$$

(1) & (2) \Rightarrow

$$C_V = 2k_B N \frac{(\epsilon\beta)^2 e^{-\beta\epsilon}}{(1 + e^{-\beta\epsilon})^2}$$

Problem 3

(a) For translational motion of indistinguishable particles the partition function had the form

$$Z_N = (Z_1)^N / N!$$

valid for $n (=N/V) \ll n_Q (=1/\lambda_{th}^3)$ or equivalently sufficiently high temperatures T and low number densities n so that quantum effects can be neglected.

(b)

$$F = -Nk_B T \ln Z_1 + k_B T (N \ln N - N)$$

$$\mu = \left(\frac{\partial F}{\partial N} \right)_{V,T}$$

$$\mu = -k_B T \ln Z_1 + k_B T [1 + \ln N - 1]$$

$$\mu = -k_B T \ln [Z_1 / N] = k_B T \ln [N / Z_1]$$

(c) For a classical ideal gas we have

$$Z_1 = V n_Q \text{ with } n_Q \text{ the quantum concentration and } n (=N/V) \ll n_Q$$

Thus we have $Z_1 / N = n_Q / n \gg 1$ or $N / Z_1 \ll 1$

Therefore, $\ln [N / Z_1] < 0$ and as a result $\mu < 0$!

(d)

$$\begin{aligned} F &= -Nk_B T \ln \left(\frac{V e}{N \lambda_{th}^3} \right) & \mu &= k_B T \ln (n \lambda_{th}^3) \\ &= Nk_B T [\ln (n \lambda_{th}^3) - 1]. \end{aligned} \quad \left. \vphantom{\begin{aligned} F &= -Nk_B T \ln \left(\frac{V e}{N \lambda_{th}^3} \right) \\ &= Nk_B T [\ln (n \lambda_{th}^3) - 1]. \end{aligned}} \right\}$$

$$\Phi_G = U - TS - \mu N = F - \mu N.$$

$$\begin{aligned} \Phi_G &= Nk_B T [\ln (n \lambda_{th}^3) - 1] - Nk_B T \ln (n \lambda_{th}^3) \\ &= -Nk_B T, \end{aligned}$$

$$pV = Nk_B T \longrightarrow \Phi_G = -pV.$$

Problem 4

(a)

$$Z_{\text{Total}} = [(Z_1)^N / N!] \times [Z_{\text{int}}]^N$$

with $Z_1 (=V/\lambda_{\text{th}}^3)$ and $Z_{\text{int}} = (1 + e^{-\beta\epsilon})^2$

(b)

Total energy will be $U_{\text{Total}} = -d \ln[(Z_1)^N / N!] / d\beta - d \ln[(Z_{\text{int}})^N] / d\beta$.

Thus, the heat capacity C_{Total} will be the sum of that due to translational motion and the internal degrees of freedom

$$C_{\text{Total}} = dU_{\text{Total}} / dT = C_{\text{trans}} + C_{\text{int}} = [3NK_B / 2] + 2NK_B [(\beta\epsilon)^2 e^{-\beta\epsilon} / (1 + e^{-\beta\epsilon})^2]$$

$C_{\text{int}} = C_V = 2NK_B [(\beta\epsilon)^2 e^{-\beta\epsilon} / (1 + e^{-\beta\epsilon})^2]$ from problem 2.

$$C_{\text{trans}} = [3NK_B / 2]$$

Problem 5

(a) The chemical potential of a photon gas in thermal equilibrium is zero, $\mu=0$, because photons are not a conserved quantity. The number of photons in a system can change freely through emission and absorption processes so that the photon gas is in thermal equilibrium.

(b)

We have

$$\ln Z = \int_0^{\infty} g(\omega) \ln(Z_{\omega}) d\omega$$

and upon substitution of Z_{ω} we obtain

$$\ln Z = \int_0^{\infty} g(\omega) \ln\left(\frac{e^{-\hbar\omega\beta/2}}{1 - e^{-\hbar\omega\beta}}\right) d\omega$$

Then

$$\ln Z = -\frac{V}{\pi^2 c^3} \int_0^{\infty} \omega^2 [\hbar\omega\beta/2] d\omega - \frac{V}{\pi^2 c^3} \int_0^{\infty} \omega^2 \ln(1 - e^{-\hbar\omega\beta}) d\omega \quad (1)$$

The first term in Eq. (1) due to the zero-point-energy is an infinite integral and we ignore it.

Thus we keep for $\ln Z$ only the finite integral (second integral) in Eq. (1). Hence, we have

$$\ln Z = -\frac{V}{\pi^2 c^3} \int_0^{\infty} \omega^2 \ln(1 - e^{-\hbar\omega\beta}) d\omega$$